N73-23363

NASA TECHNICAL NOTE

NASA TN D-7284

NASA TN D-7284

NONLINEAR EVOLUTION OF A DISTURBANCE IN AN UNBOUNDED VISCOUS FLUID WITH UNIFORM SHEAR

by Robert G. Deissler and Burt M. Rosenbaum

Lewis Research Center
Cleveland, Ohio 44135

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1973

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TN D-7284		

4. Title and Subtitle	5. Report Date
NONLINEAR EVOLUTION OF A DISTURBANCE IN AN UNBOUNDED VISCOUS FLUID WITH UNIFORM SHEAR	May 1973
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
Robert G. Deissler and Burt M. Rosenbaum	E-6857
	10. Work Unit No.
9. Performing Organization Name and Address	502-04
Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135	11. Contract or Grant No.
12. Sponsoring Agency Name and Address	13. Type of Report and Period Covered
National Aeronautics and Space Administration Washington, D.C. 20546	Technical Note
	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The evolution of a disturbance in the presence of a uniform mean velocity gradient is calculated by a power-series solution of the incompressible Navier-Stokes equations. Terms through those in time cubed are retained in the solution. For the initial condition a three-dimensional cosine distribution with two harmonic terms is assumed. The nonlinear interaction of these harmonic terms produces new harmonics which in turn interact. For large velocity gradients the energy of the disturbance can grow with time. The results may shed some light on the maintenance or growth of turbulence in a shear flow.

17. Key Words (Suggested by Author(s))		18. Distribution Statement
Nonlinear	Mean shear	Unclassified - unlimited
Disturbance	Turbulence	
Viscous fluid		

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	23	$3.00

* For sale by the National Technical Information Service, Springfield, Virginia 22151

NONLINEAR EVOLUTION OF A DISTURBANCE IN AN UNBOUNDED VISCOUS FLUID WITH UNIFORM SHEAR

by Robert G. Deissler and Burt M. Rosenbaum

Lewis Research Center

SUMMARY

The evolution of a disturbance in the presence of a uniform mean velocity gradient is calculated by a power-series solution of the incompressible Navier-Stokes equations. Terms through those in time cubed are retained in the solution. For the initial condition a three-dimensional cosine distribution with two harmonic terms is assumed. The nonlinear interaction of these harmonic terms produces new harmonics which in turn interact. For large velocity gradients the energy of the disturbance can grow with time. The results may shed some light on the maintenance or growth of turbulence in a shear flow.

INTRODUCTION

The history of a disturbance in a viscous fluid has been calculated to various degrees of approximation for several initial conditions in references 1 to 3. The results showed that the evolution of the disturbance is modified by the generation of new harmonics through the interaction of those already present. The effect of chemical reaction on a fluid disturbance was studied in reference 4. Various numerical aspects of Fourier-type solutions of the Navier-Stokes initial value problem were considered in reference 5.

In the present report the results of reference 3 are extended to include the effect of a uniform velocity gradient on the evolution of a fluid disturbance. Thus, in addition to mode-mode interactions (interactions between the Fourier components of the wave), interactions between the velocity fluctuations and a mean gradient are considered. As in reference 3 the initial condition for the velocity disturbance is taken to consist of two waves with different wave number and intensity vectors. It is noted that turbulence can be regarded as being made up of a very large number of disturbances of the type considered in this report, so that the results may shed some light on turbulent shear flow,

particularly in the early stages of development. In particular, it would seem that nonlinear effects might be more easily investigated in the present problem than in turbulence. As will be mentioned in the section RESULTS AND DISCUSSION, there are, however, important differences between the results obtained for simple regular disturbances and those for a fully developed random turbulence. The case of turbulence with a uniform shear, but without nonlinear effects, is studied in reference 6.

For simplicity a Taylor power series in time rather than the exponential method of reference 3 is used. The basic equations and their analysis are considered in the next section.

ANALYSIS

The following problem is considered: given the initial velocity distribution in an unbounded viscous fluid with a uniform mean velocity gradient, predict the motion at later times. For a viscous fluid with constant properties, the three-dimensional equations of motion in dimensionless form are

$$\frac{\partial \tilde{u}_i}{\partial t} = -\frac{\partial p}{\partial \bar{x}_i} - \frac{\partial(\tilde{u}_i \tilde{u}_k)}{\partial \bar{x}_k} + \frac{\partial^2 \tilde{u}_i}{\partial \bar{x}_k \, \partial \bar{x}_k} \tag{1}$$

and

$$\frac{\partial^2 p}{\partial \bar{x}_l \, \partial \bar{x}_l} = -\frac{\partial^2(\tilde{u}_l \tilde{u}_k)}{\partial \bar{x}_l \, \partial \bar{x}_k} \tag{2}$$

where equation (2) is obtained by taking the divergence of equation (1) and applying the continuity equation, and

$$\tilde{u}_i = \frac{x_0}{\nu} u_i^*$$

$$p = \frac{x_0^2}{\rho \nu^2} p^*$$

$$t = \frac{\nu}{x_0^2} t^*$$

$$\bar{x}_i = \frac{\vec{x}_i^*}{x_0}$$

The quantity u_i^* is a velocity component, $\vec{x}_i^*$ is a space coordinate, x_0 is a characteristic length, t^* is the time, ρ is the constant density, and ν is the constant kinematic viscosity. Note that the stars on dimensional quantities are omitted for corresponding dimensionless quantities. The subscripts in equations (1) and (2) can take on the values 1, 2, and 3, and a repeated subscript in a term indicates a summation. (Symbols are defined in the appendix.)

One can break the dimensionless instantaneous velocities into mean and spatially fluctuating components; thus, set $\tilde{u}_i = U_i + u_i$. Equations (1) and (2) become

$$\left(\frac{\partial}{\partial t} + U_k \frac{\partial}{\partial \bar{x}_k}\right) u_i = -\frac{\partial U_i}{\partial t} - \frac{\partial p}{\partial \bar{x}_i} - \frac{\partial u_i u_k}{\partial \bar{x}_k} - u_k \frac{\partial U_i}{\partial \bar{x}_k} - \frac{\partial U_i U_k}{\partial \bar{x}_k} + \frac{\partial^2 u_i}{\partial \bar{x}_k \partial \bar{x}_k} + \frac{\partial^2 U_i}{\partial \bar{x}_k \partial \bar{x}_k} \tag{3}$$

$$\frac{\partial^2 p}{\partial \bar{x}_l \partial \bar{x}_l} = -\frac{\partial^2 u_l u_k}{\partial \bar{x}_l \partial \bar{x}_k} - 2 \frac{\partial u_l}{\partial \bar{x}_k} \frac{\partial U_k}{\partial \bar{x}_l} - \frac{\partial^2 U_l U_k}{\partial \bar{x}_l \partial \bar{x}_k} \tag{4}$$

If the mean velocity is assumed to be in the x_1-direction, the left side of equation (3) becomes $[\partial/\partial t + U_1(\partial/\partial x_1)] u_i$. In this analysis we let the observer move with the mean velocity at each point, since we are interested in the changes with time of fluctuation levels from that point of view. Also, the mean velocity gradient is taken to be in the x_2-direction, constant in time and space, and equal to $dU_1/d\bar{x}_2 \equiv S$. Coordinates x_i relative to the moving observer are given by the transformation $x_i = \bar{x}_i - \delta_{i1} U_1(\bar{x}_2)t$. Then

$$\frac{\partial}{\partial \bar{x}_i} = \frac{\partial}{\partial x_i} - \delta_{2i} St \frac{\partial}{\partial x_1}$$

and

$$\frac{\partial^2}{\partial \bar{x}_i \partial \bar{x}_j} = \frac{\partial^2}{\partial x_i \partial x_j} - \delta_{2j} St \frac{\partial^2}{\partial x_1 \partial x_i} - \delta_{2i} St \frac{\partial^2}{\partial x_1 \partial x_j} + \delta_{2i}\delta_{2j} S^2 t^2 \frac{\partial^2}{\partial x_1^2}$$

Also,

$$\left(\frac{\partial}{\partial t}\right)_{\bar{x}_l} = \left(\frac{\partial}{\partial t}\right)_{x_l} - U_1(\bar{x}_2)\frac{\partial}{\partial x_1}$$

Equations (3) and (4) become, in the new coordinate system,

$$\frac{\partial u_i}{\partial t} = -\delta_{i1} S u_2 - \frac{\partial u_i u_k}{\partial x_k} - \frac{\partial p}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_k \partial x_k} + St\frac{\partial u_i u_2}{\partial x_1} + \delta_{i2} St\frac{\partial p}{\partial x_1} - 2St\frac{\partial^2 u_i}{\partial x_1 \partial x_2} + S^2 t^2 \frac{\partial^2 u_i}{\partial x_1^2} \tag{5}$$

$$\frac{\partial^2 p}{\partial x_l \partial x_l} = -2S\frac{\partial u_2}{\partial x_1} - \frac{\partial^2 u_l u_k}{\partial x_l \partial x_k} + 2St\frac{\partial^2 p}{\partial x_1 \partial x_2} + 2St\frac{\partial^2 u_l u_2}{\partial x_1 \partial x_l} - S^2 t^2 \frac{\partial^2 u_2^2}{\partial x_1^2} - S^2 t^2 \frac{\partial^2 p}{\partial x_1^2} \tag{6}$$

where δ_{ij} is the Kronecker delta. In addition, the equation of continuity which had the form

$$\frac{\partial u_i}{\partial \bar{x}_i} = 0$$

now becomes

$$\frac{\partial u_i}{\partial x_i} = St\frac{\partial u_2}{\partial x_1} \tag{7}$$

For the initial condition on the velocity fluctuation at time $t = 0$, take

$$\left(u_i\right)_0 = \sum_{m'} a_i^{m'} \cos \vec{q}^{m'} \cdot \vec{x} \tag{8}$$

where m' can, in the general case, be any positive integer, and $a_i^{m'}$ and $q_i^{m'}$ are, respectively, intensity and wave-number vectors. In the actual numerical calculations m' will be set equal to 1, 2. Two is, of course, the smallest number of terms that can be retained if there are to be mode-mode interactions.

To get the evolution of the velocity fluctuation, a Taylor series in time is used and the required initial time derivatives are obtained from equations (5), (6), and (8). The Taylor series is

$$u_i = \left(u_i\right)_0 + \left(\frac{\partial u_i}{\partial t}\right)_0 t + \left(\frac{\partial^2 u_i}{\partial t^2}\right)_0 \frac{t^2}{2} + \left(\frac{\partial^3 u_i}{\partial t^3}\right)_0 \frac{t^3}{6} + \dots \tag{9}$$

The quantity $\left(u_i\right)_0$ in this equation is obtained from equation (8), which can be more conveniently written in complex notation as

$$\left(u_i\right)_0 = \sum_m \frac{1}{2} a_i^m e^{i\vec{q}^m \cdot \vec{x}} \tag{10}$$

where m takes on both positive and negative values (i.e., $m = \pm m'$), and $q_i^{-m} = -q_i^m$ and $a_i^{-m} = a_i^m$. In the case considered in this report, where $m' = 1, 2$, the values m takes on are given by $m = -2, -1, 1, 2$. Also, the equation of continuity yields the relation

$$a_i^m q_i^m = 0 \tag{11}$$

(Note that, although a repeated subscript indicates a summation, summations over superscripts are only to be carried out when explicitly indicated by the summation sign.)

To obtain $\left(\partial u_i/\partial t\right)_0$ in equation (9), equation (10) is first substituted into equation (6) evaluated at the initial time. This gives

$$\frac{\partial^2 p_0}{\partial x_l \, \partial x_l} = -iS \sum_m a_2^m q_1^m e^{i\vec{q}^m \cdot \vec{x}} + \sum_{m,n} \frac{1}{4} a_l^m a_k^n q_l^n q_k^m e^{\left(\vec{q}^m + \vec{q}^n\right) \cdot \vec{x}} \tag{12}$$

where m and n are independently assigned the values assigned to m. Inspection of equation (12) shows that p_0 has the form

$$p_0 = i \sum_m \pi^m e^{i\vec{q}^m \cdot \vec{x}} + \sum_{m,n} \pi^{mn} e^{i\left(\vec{q}^m + \vec{q}^n\right) \cdot \vec{x}} \tag{13}$$

Substituting equations (13) into equation (12) and equating coefficients of like powers of e give

$$\pi^m = S\frac{a_2^m q_1^m}{q_l^m q_l^m} \tag{14a}$$

$$\pi^{mn} = -\frac{1}{4}\frac{a_l^m a_k^n q_l^n q_k^m}{\left(q_j^m + q_j^n\right)\left(q_j^m + q_j^n\right)} \tag{14b}$$

where equation (11) has been used to simplify the expression for π^{mn}. Substituting equations (10) and (13) into equation (5) evaluated at $t = 0$ gives for the first time derivative in equation (9)

$$\left(\frac{\partial u_i}{\partial t}\right)_0 = \sum_m b_i^m e^{i\vec{q}^m\cdot\vec{x}} + i\sum_{m,n} a_i^{mn} e^{i(\vec{q}^m+\vec{q}^n)\cdot\vec{x}} \tag{15}$$

where

$$b_i^m = \pi^m q_i^m - \frac{1}{2}\delta_{i1}Sa_2^m - \frac{1}{2}a_i^m q_k^m q_k^m \tag{16a}$$

$$a_i^{mn} = -\pi^{mn}\left(q_i^m + q_i^n\right) - \frac{1}{8}\left(a_i^m a_k^n q_k^m + a_i^n a_k^m q_k^n\right) \tag{16b}$$

and a_i^{mn} has been written so as to maintain symmetry in the superscripts m and n.

To obtain $\left(\partial^2 u_i/\partial t^2\right)_0$ in equation (9), differentiate equations (5) and (6) with respect to time. This gives at $t = 0$

$$\left(\frac{\partial^2 u_i}{\partial t^2}\right)_0 = -\delta_{i1}S\left(\frac{\partial u_2}{\partial t}\right)_0 - \frac{\partial}{\partial x_k}\left[\left(u_i\right)_0\left(\frac{\partial u_k}{\partial t}\right)_0 + \left(u_k\right)_0\left(\frac{\partial u_i}{\partial t}\right)_0\right] - \frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial t}\right)_0 + \frac{\partial^2}{\partial x_k\,\partial x_k}\left(\frac{\partial u_i}{\partial t}\right)_0$$
$$+ S\left\{\left[\frac{\partial(u_i u_2)}{\partial x_1}\right]_0 + \delta_{i2}\left(\frac{\partial p}{\partial x_1}\right)_0 - 2\left(\frac{\partial^2 u_i}{\partial x_1\,\partial x_2}\right)_0\right\} \tag{17}$$

where $(\partial p/\partial t)_0$ is given by

$$\frac{\partial^2}{\partial x_l\,\partial x_l}\left(\frac{\partial p}{\partial t}\right)_0 = -2S\frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial t}\right)_0 - 2\frac{\partial^2}{\partial x_l\,\partial x_k}\left[\left(u_l\right)_0\left(\frac{\partial u_k}{\partial t}\right)_0\right] + 2S\left\{\left(\frac{\partial^2 p}{\partial x_1\,\partial x_2}\right)_0 + \left[\frac{\partial^2(u_l u_2)}{\partial x_1\,\partial x_2}\right]_0\right\} \tag{18}$$

Substituting equations (10), (13), and (15) into equations (18) and (17) and proceeding as was done to obtain $\left(\partial u_i/\partial t\right)_0$ give

$$\left(\frac{\partial^2 u_i}{\partial t^2}\right)_0 = \sum_m c_i^m e^{i\vec{q}^m\cdot\vec{x}} + i\sum_{m,n} b_i^{mn} e^{i(\vec{q}^m+\vec{q}^n)\cdot\vec{x}} + \sum_{m,n,r} a_i^{mnr} e^{i(\vec{q}^m+\vec{q}^n+\vec{q}^r)\cdot\vec{x}} \tag{19}$$

where

$$c_i^m = \alpha^m q_i^m - b_i^m q_k^m q_k^m - \delta_{i1} S b_2^m - \delta_{i2} S\pi^m q_1^m + S a_i^m q_1^m q_2^m \tag{20a}$$

$$b_i^{mn} = -\alpha^{mn}\left(q_i^m + q_i^n\right) - \frac{1}{4}\left(a_i^m b_k^n q_k^m + a_i^n b_k^m q_k^n + a_k^m b_i^n q_k^n + a_k^n b_i^m q_k^m\right)$$

$$- a_i^{mn}\left(q_k^m + q_k^n\right)\left(q_k^m + q_k^n\right) - \delta_{i1} S a_2^{mn} + \delta_{i2} S\pi^{mn}\left(q_1^m + q_1^n\right) + \frac{1}{8} S\left(a_i^n a_2^m q_1^n + a_i^m a_2^n q_1^m\right) \tag{20b}$$

$$a_i^{mnr} = \alpha^{mnr}\left(q_i^m + q_i^n + q_i^r\right) + \frac{1}{6}\left[a_i^m a_k^{nr} q_k^m + a_i^n a_k^{rm} q_k^n + a_i^r a_k^{mn} q_k^r + a_k^m a_i^{nr}\left(q_k^n + q_k^r\right)\right.$$

$$\left. + a_k^n a_i^{mr}\left(q_k^m + q_k^r\right) + a_k^r a_i^{mn}\left(q_k^m + q_k^n\right)\right] \tag{20c}$$

ıd where

$$\alpha^m q_l^m q_l^m = 2S\pi^m q_1^m q_2^m + 2S b_2^m q_1^m \tag{21a}$$

$$\alpha^{mn}\left(q_l^m + q_l^n\right)\left(q_l^m + q_l^n\right) = 2S\pi^{mn}\left(q_1^m + q_1^n\right)\left(q_2^m + q_2^n\right) - \frac{1}{2}\left(a_l^m b_k^n q_l^n q_k^m + a_l^n b_k^m q_l^m q_k^n\right)$$

$$- 2Sa_2^{mn}\left(q_1^m + q_1^n\right) + \frac{1}{4} S\left(a_2^m a_k^n q_k^m q_1^n + a_2^n a_k^m q_k^n q_1^m\right) \tag{21b}$$

$$\alpha^{mnr}\left(q_l^m + q_l^n + q_l^r\right)\left(q_l^m + q_l^n + q_l^r\right) = -\frac{1}{3}\left[a_l^m a_k^{nr} q_k^m\left(q_l^n + q_l^r\right) + a_l^n a_k^{mr} q_k^n\left(q_l^m + q_l^r\right)\right.$$

$$\left. + a_l^r a_k^{mn} q_k^r\left(q_l^m + q_l^n\right)\right] \tag{21c}$$

Equation (11) and the relations

$$b_i^m q_i^m = \frac{1}{2} Sa_2^m q_i^m$$

$$a_i^{mn}\left(q_i^m + q_i^n\right) = 0$$

have been used to simplify equations (20) and (21). These relations were obtained by differentiating equation (7) with respect to time t, setting $t = 0$, and then substituting the expressions given by equations (8) and (15) for $\left(u_2\right)_0$ and $\left(\partial u_i/\partial t\right)_0$, respectively.

Similarly,

$$\left(\frac{\partial^3 u_i}{\partial t^3}\right)_0 = \sum_m d_i^m e^{i\vec{q}^m \cdot \vec{x}} + i \sum_{m,n} c_i^{mn} e^{i(\vec{q}^m + \vec{q}^n)\cdot \vec{x}} + \sum_{m,n,r} b_i^{mnr} e^{i(\vec{q}^m + \vec{q}^n + \vec{q}^r)\cdot \vec{x}}$$

$$+ i \sum_{m,n,r,s} a_i^{mnrs} e^{i(\vec{q}^m + \vec{q}^n + \vec{q}^r + \vec{q}^s)\cdot \vec{x}} \tag{22}$$

where

$$d_i^m = \beta^m q_i^m - c_i^m q_k^m q_k^m - \delta_{i1} Sc_2^m - 2\delta_{i2} S\alpha^m q_1^m + 4Sb_i^m q_1^m q_2^m - S^2 a_i^m q_1^m q_1^m \tag{23a}$$

$$c_i^{mn} = -\beta^{mn}\left(q_i^m + q_i^n\right) - \frac{1}{4}\left(a_i^m c_k^n q_k^m + a_i^n c_k^m q_k^n + a_k^m c_i^n q_k^n + a_k^n c_i^m q_k^m\right) - \left(b_i^m b_k^n q_k^m + b_i^n b_k^m q_k^n\right)$$

$$- b_i^{mn}\left(q_k^m + q_k^n\right)\left(q_k^m + q_k^n\right) - \delta_{i1} S b_2^{mn} + 2\delta_{i2} S \alpha^{mn}\left(q_1^m + q_1^n\right)$$

$$+ \frac{1}{2} S\left(a_2^m b_i^n q_1^n + a_2^n b_i^m q_1^m + a_i^m b_2^n q_1^m + a_i^n b_2^m q_1^n\right) + 4 S a_i^{mn}\left(q_1^m + q_1^n\right)\left(q_2^m + q_2^n\right) \tag{23b}$$

$$b_i^{mnr} = \beta^{mnr}\left(q_i^m + q_i^n + q_i^r\right) + \frac{1}{6}\Big[a_i^m b_k^{nr} q_k^m + a_i^n b_k^{mr} q_k^n + a_i^r b_k^{mn} q_k^r + a_k^m b_i^{nr}\left(q_k^n + q_k^r\right)$$

$$+ a_k^n b_i^{mr}\left(q_k^m + q_k^r\right) + a_k^r b_i^{mn}\left(q_k^m + q_k^n\right)\Big] + \frac{2}{3}\Big[a_k^{mn} b_i^r q_k^r + a_k^{mr} b_i^n q_k^n + a_k^{nr} b_i^m q_k^m$$

$$+ a_i^{mn} b_k^r\left(q_k^m + q_k^n\right) + a_i^{mr} b_k^n\left(q_k^m + q_k^r\right) + a_i^{nr} b_k^m\left(q_k^n + q_k^r\right)\Big]$$

$$- a_i^{mnr}\left(q_k^m + q_k^n + q_k^r\right)\left(q_k^m + q_k^n + q_k^r\right) - \delta_{i1} S a_2^{mnr} - 2\delta_{i2} S \alpha^{mnr}\left(q_1^m + q_1^n + q_1^r\right)$$

$$- \frac{1}{3} S\Big[a_2^m a_i^{nr}\left(q_1^n + q_1^r\right) + a_2^n a_i^{mr}\left(q_1^m + q_1^r\right) + a_2^r a_i^{mn}\left(q_1^m + q_1^n\right)$$

$$+ a_i^m a_2^{nr} q_1^m + a_i^n a_2^{mr} q_1^n + a_i^r a_2^{mn} q_1^r\Big] \tag{23c}$$

$$a_i^{mnrs} = -\beta^{mnrs}\left(q_i^m + q_i^n + q_i^r + q_i^s\right) - \frac{1}{8}\Big[a_i^m a_k^{nrs} q_k^m + a_i^n a_k^{mrs} q_k^n$$

$$+ a_i^r a_k^{mns} q_k^r + a_i^s a_k^{mnr} q_k^s + a_k^m a_i^{nrs}\left(q_k^n + q_k^r + q_k^s\right) + a_k^n a_i^{mrs}\left(q_k^m + q_k^r + q_k^s\right)$$

$$+ a_k^r a_i^{mns}\left(q_k^m + q_k^n + q_k^s\right) + a_k^s a_i^{mnr}\left(q_k^m + q_k^n + q_k^r\right)\Big] + \frac{1}{3}\Big[a_i^{mn} a_k^{rs}\left(q_k^m + q_k^n\right)$$

$$+ a_i^{mr} a_k^{ns}\left(q_k^m + q_k^r\right) + a_i^{ms} a_k^{nr}\left(q_k^m + q_k^s\right) + a_k^{mn} a_i^{rs}\left(q_k^r + q_k^s\right) + a_k^{mr} a_i^{ns}\left(q_k^n + q_k^s\right)$$

$$+ a_k^{ms} a_i^{nr}\left(q_k^n + q_k^r\right)\Big] \tag{23d}$$

and where

$$\beta^{m} q_{l}^{m} q_{l}^{m} = 4S\alpha^{m} q_{1}^{m} q_{2}^{m} + 2Sc_{2}^{m} q_{1}^{m} - 2S^{2}\pi^{m} q_{1}^{m} q_{1}^{m} \tag{24a}$$

$$\begin{aligned}
\beta^{mn}\left(q_{l}^{m} + q_{l}^{n}\right)\left(q_{l}^{m} + q_{l}^{n}\right) = {} & 4S\alpha^{mn}\left(q_{1}^{m} + q_{1}^{n}\right)\left(q_{2}^{m} + q_{2}^{n}\right) - 2S^{2}\pi^{mn}\left(q_{1}^{m} + q_{1}^{n}\right)\left(q_{1}^{m} + q_{1}^{n}\right) \\
& - \frac{1}{2}\left(a_{l}^{m} c_{k}^{n} q_{l}^{n} q_{k}^{m} + a_{l}^{n} c_{k}^{m} q_{l}^{m} q_{k}^{n}\right) - 2b_{l}^{m} b_{k}^{n} q_{l}^{n} q_{k}^{m} - \frac{1}{2} S^{2} a_{2}^{m} a_{2}^{n} q_{1}^{m} q_{1}^{n} \\
& - 2Sb_{2}^{mn}\left(q_{1}^{m} + q_{1}^{n}\right) + S\left(a_{2}^{m} b_{k}^{n} q_{k}^{m} q_{1}^{n} + a_{2}^{n} b_{k}^{m} q_{k}^{n} q_{1}^{m}\right. \\
& \left. + a_{k}^{m} b_{2}^{n} q_{k}^{n} q_{1}^{m} + a_{k}^{n} b_{2}^{m} q_{k}^{m} q_{1}^{n}\right)
\end{aligned} \tag{24b}$$

$$\begin{aligned}
\beta^{mnr}\left(q_{l}^{m} + q_{l}^{n} + q_{l}^{r}\right)\left(q_{l}^{m} + q_{l}^{n} + q_{l}^{r}\right) = {} & 4S\alpha^{mnr}\left(q_{1}^{m} + q_{1}^{n} + q_{1}^{r}\right)\left(q_{2}^{m} + q_{2}^{n} + q_{2}^{r}\right) \\
& - \frac{1}{3}\left[a_{l}^{m} b_{k}^{nr} q_{k}^{m}\left(q_{l}^{n} + q_{l}^{r}\right) + a_{l}^{n} b_{k}^{mr} q_{k}^{n}\left(q_{l}^{m} + q_{l}^{r}\right)\right. \\
& \left. + a_{l}^{r} b_{k}^{mn} q_{k}^{r}\left(q_{l}^{m} + q_{l}^{n}\right)\right] - \frac{4}{3}\left[a_{l}^{mn} b_{k}^{r} q_{l}^{r}\left(q_{k}^{m} + q_{k}^{n}\right)\right. \\
& \left. + a_{l}^{mr} b_{k}^{n} q_{l}^{n}\left(q_{k}^{m} + q_{k}^{r}\right) + a_{l}^{nr} b_{k}^{m} q_{l}^{m}\left(q_{k}^{n} + q_{k}^{r}\right)\right] \\
& + 2Sa_{2}^{mnr}\left(q_{1}^{m} + q_{1}^{n} + q_{1}^{r}\right) + \frac{2}{3} S\left[a_{2}^{m} a_{k}^{nr} q_{k}^{m}\left(q_{1}^{n} + q_{1}^{r}\right)\right. \\
& + a_{2}^{n} a_{k}^{mr} q_{k}^{n}\left(q_{1}^{m} + q_{1}^{r}\right) + a_{2}^{r} a_{k}^{mn} q_{k}^{r}\left(q_{1}^{m} + q_{1}^{n}\right) \\
& + a_{k}^{m} a_{2}^{nr} q_{1}^{m}\left(q_{k}^{n} + q_{k}^{r}\right) + a_{k}^{n} a_{2}^{mr} q_{1}^{n}\left(q_{k}^{m} + q_{k}^{r}\right) \\
& \left. + a_{k}^{r} a_{2}^{mn} q_{1}^{r}\left(q_{k}^{m} + q_{k}^{n}\right)\right]
\end{aligned} \tag{24c}$$

$$\beta^{mnrs}\left(q_l^m + q_l^n + q_l^r + q_l^s\right)\left(q_l^m + q_l^n + q_l^r + q_l^s\right) = -\frac{1}{4}\left[a_l^m a_k^{nrs} q_k^m \left(q_l^n + q_l^r + q_l^s\right)\right.$$

$$+ a_l^n a_k^{mrs} q_k^n \left(q_l^m + q_l^r + q_l^s\right)$$

$$+ a_l^r a_k^{mns} q_k^r \left(q_l^m + q_l^n + q_l^s\right)$$

$$\left.+ a_l^s a_k^{mnr} q_k^s \left(q_l^m + q_l^n + q_l^r\right)\right]$$

$$+ \frac{2}{3}\left[a_l^{mn} a_k^{rs}\left(q_l^r + q_l^s\right)\left(q_k^m + q_k^n\right)\right.$$

$$+ a_l^{mr} a_k^{ns}\left(q_l^n + q_l^s\right)\left(q_k^m + q_k^r\right)$$

$$\left.+ a_l^{ms} a_k^{nr}\left(q_l^n + q_l^r\right)\left(q_k^m + q_k^s\right)\right] \tag{24d}$$

The additional relations obtained from differentiating the equation of continuity and employed to simplify equations (23) and (24) are

$$c_i^m q_i^m = 2Sb_2^m q_i^m$$

$$b_i^{mn}\left(q_i^m + q_i^n\right) = 2Sa_2^{mn}\left(q_1^m + q_1^n\right)$$

$$a_i^{mnr}\left(q_i^m + q_i^n + q_i^r\right) = 0$$

Substituting equations (10), (15), (16), and (19) to (24) into equation (8) gives the time evolution of u_i as a function of the coordinates in the moving system for small and moderate times. However, we are more interested in the space-averaged value $\overline{u_i u_j}$. If the product $u_i u_j$ is averaged (integrated) over space, the integrals of the various terms are zero except for those for which the exponential factor reduces to 1, so that the result is, for terms up to order t^3,

$$\overline{u_i u_j} = A_{ij} + B_{ij}t + C_{ij}t^2 + D_{ij}t^3 \tag{25}$$

where the coefficients of the powers of t are given by

$$A_{ij} = \frac{1}{4} \sum_{m} a_i^m a_j^m \tag{26a}$$

$$B_{ij} = \frac{1}{2} \sum_{m} \left(a_i^m b_j^m + a_j^m b_i^m\right) \tag{26b}$$

$$C_{ij} = \sum_{m} \left(\frac{a_i^m c_j^m + a_j^m c_i^m}{4} + b_i^m b_j^m\right) + \sum_{\substack{m,n,r,s \\ (m+n+r+s=0)}} \left(\frac{a_i^m a_j^{nrs} + a_j^m a_i^{nrs}}{4} - a_i^{mn} a_j^{rs}\right) \tag{26c}$$

$$D_{ij} = \sum_{m} \left(\frac{a_i^m d_j^m + a_j^m d_i^m}{12} + \frac{b_i^m c_j^m + b_j^m c_i^m}{2}\right)$$

$$+ \sum_{\substack{m,n,r,s \\ (m+n+r+s=0)}} \left(\frac{a_i^m b_j^{nrs} + a_j^m b_i^{nrs}}{12} + \frac{b_i^m a_j^{nrs} + b_j^m a_i^{nrs}}{2} - \frac{a_i^{mn} b_j^{rs} + a_j^{mn} b_i^{rs}}{2}\right) \tag{26d}$$

Although $\overline{u_i u_j}$ has been obtained by integrating $u_i u_j$ with respect to coordinates in the moving system, its value is not dependent on carrying out the integration in that system and will be the same in the stationary system.

RESULTS AND DISCUSSION

As in the case of nonlinear decay without a mean velocity gradient (ref. 3), the nonlinear terms in the Navier-Stokes equations here produce a proliferation of a new eddies or harmonic components at various wave numbers. These effects are contained in the

double, triple, and quadruple summations in equations (15), (19), and (22). In the present case the presence of terms containing mean gradients produces harmonic components in addition to those in reference 3. For instance, in equation (17), the nonlinear term $u_i\left(\partial u_k/\partial t\right)_0$ contains mean gradient effects through equations (15) and (16).

Figure 1 compares the results of the present analysis for $dU_1/d\bar{x}_2 = 0$ with those from reference 3 for the same initial conditions. Note that these initial conditions satisfy continuity and give nonzero interaction terms, even for $S = dU_1/d\bar{x}_2 = 0$ $\left(a_i^1 q_i^1 = a_i^2 q_i^2 = 0,\ a_i^1 q_i^2 \neq 0, \text{ and } a_i^2 q_i^1 \neq 0\right)$. For $S = 0$, the only essential difference between the two analyses is that the present analysis utilizes a Taylor series, whereas that in reference 3 results in a series of exponentials. The first, second, and third approximations for the Taylor series correspond, respectively, to retaining terms in t, t^2, and t^3 in equation (25). Comparison of the various approximations for the Taylor series solution and of the results for the Taylor series with those for the exponential method should give an idea of the accuracy of the results.

The results in figure 1 for the present Taylor series solution appear to be accurate to a dimensionless time of about 0.003, whereas those for the exponential method are good for much larger times. The Taylor series should still give a good indication of the effect of shear on a disturbance, particularly for large shear, where large effects of shear at small times might be expected.

For one accustomed to thinking in terms of turbulence, it may seem surprising that the shear component $\overline{u_1 u_2}$ is not zero for a zero mean velocity gradient. It should be recalled, however, that the present disturbance is for a limited number of wave-number vectors. Even for turbulence, it is necessary to integrate over all directions in wave-number space to obtain zero shear stress for zero velocity gradient.

The effect of shear on the kinetic energy $\overline{u_i u_i}/2$ is shown in figure 2, where $\overline{u_i u_i}/2$ is plotted against dimensionless time for a low, an intermediate, and a high velocity gradient. (The same initial conditions are used for figs. 1 to 4.) For the low velocity gradient $S = 20$, the energy decays, as for zero velocity gradient. For the intermediate velocity gradient, the energy decays at a slow rate at small times and then appears to increase. For the large velocity gradient the energy in figure 2 increases monotonically.

Figure 3 shows the behavior of the individual components of the disturbance for a high velocity gradient. All the components except $\overline{u_3^2}$ increase with time. Inasmuch as production terms (terms multiplied by δ_{i1}) do not occur in the equations for $\overline{u_2^2}$, one might suppose that component increases because of energy transferred into it from $\overline{u_1^2}$ by pressure terms (terms with q_i^m in the denominator). Figure 4, where the pressure terms have been neglected, indicates that is the case, since $\overline{u_2^2}$ without the pressure terms decreases with time. The shear component still increases, apparently because the u_1 in $-\overline{u_1 u_2}$ increases sufficiently rapidly to offset the effect of the decrease in u_2.

All of the results thus far were for the same initial conditions. In order to get an idea of how changes in initial conditions can affect the components of the disturbance, the directions of the intensity and wave-number vectors relative to the direction of the velocity gradient are varied in figures 5(a) to (e). Those figures, together with figure 3, present the results for a set of six permutations (out of a possible 36) of the components of the intensity and wave-number vectors for fixed magnitudes of the vectors. Comparison of the various figures shows that changes in the initial conditions affect the evolution of the disturbance both quantitatively and qualitatively.

Perhaps the most interesting aspect of the curves in figure 5 is that, although the equations for $\overline{u_1^2}$ contain production terms, $\overline{u_1^2}$ for $dU_1/d\bar{x}_2 \neq 0$ decays faster than it does for $dU_1/d\bar{x}_2 = 0$ in several cases. That result might be caused by a change in sign of the production term, in which case the shear component $\overline{u_1u_2}$ would become positive, or by large energy transfer out of $\overline{u_1^2}$ into the other components by the pressure terms. The results show that both of these effects can be important. In figures 5(a) and (d) $\overline{u_1u_2}$ does change sign and become positive, so that the production term becomes negative. In that case $\overline{u_1u_2}$ tends to increase S rather than to decrease it, as it normally does for turbulence. That is, it acts like a pump rather than a brake on the fluid. There is also an effect of pressure terms in transferring energy into $\overline{u_2^2}$ from other components, since $\overline{u_2^2}$ increases with time even though the equations for that component do not contain a production term.

In figures 5(b) and (c), on the other hand, the abnormally high decay rate of $\overline{u_1^2}$ is due entirely to pressure forces since $\overline{u_1u_2}$ is negative and thus the production term for $\overline{u_1^2}$ is positive in those figures. The large effect of directional energy transfer in those figures is also shown by the fact that both $\overline{u_2^2}$ and $\overline{u_3^2}$ increase with time, even though the equations for those components do not contain production terms.

CONCLUDING REMARKS

The nonlinear terms in the Navier-Stokes equations produce a proliferation of new harmonic disturbances at various wave numbers. The presence of a mean gradient produces harmonic components in addition to those produced when it is absent. Although the present Taylor series solution seems to be limited to shorter times than is an analysis in which exponentials are obtained, it is capable of showing the effects of a mean velocity gradient on the early evolution of a disturbance. As the mean gradient increases, the rate of decay of the kinetic energy of the disturbance decreases. For large gradients the energy can increase with time. However, for the chosen initial conditions, at least one of the directional components always decayed.

Rotating the intensity and wave-number vectors of the initial disturbance changed the evolution pattern of the disturbance both quantitatively and qualitatively. For some

orientations the shear component of the disturbance changed sign, so that it tended to increase the velocity gradient (pump the fluid), rather than to decrease it (brake the fluid). Also, the shear components of the disturbance could be nonzero even when the velocity gradient was zero. These results differ from those for homogeneous turbulence apparently because the disturbance considered in this report consisted of a limited number of harmonic components. As in turbulence, the pressure forces played a significant role in the directional distribution of the disturbance energy.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, March 13, 1973,
502-04.

APPENDIX - SYMBOLS

a_i^m	defined by eq. (8) or (10)
a_i^{mn}	defined by eq. (16b)
a_i^{mnr}	defined by eq. (20c)
a_i^{mnrs}	defined by eq. (23d)
b_i^m	defined by eq. (16a)
b_i^{mn}	defined by eq. (20b)
b_i^{mnr}	defined by eq. (23c)
c_i^m	defined by eq. (20a)
c_i^{mn}	defined by eq. (23b)
d_i^m	defined by eq. (23a)
p	dimensionless pressure, $\left(x_0^2/\rho\nu^2\right)p^*$
p^*	pressure
q_i^m	component of initial dimensionless wave number vector $x_0 q_i^{m*}$
q_i^{m*}	component of initial wave number vector
$\vec{q}^m$	initial dimensionless wave number vector $x_0 q^{m*}$
$\vec{q}^{m*}$	initial wave number vector
S	dimensionless mean-velocity gradient, $dU_1/d\bar{x}_2$
t	dimensionless time, $\nu t^*/x_0^2$
t^*	time
U_i	component of mean velocity
u_i	component of dimensionless spatially fluctuating velocity
$\tilde{u}_i$	component of dimensionless velocity, $x_0 u_i^*/\nu$
u_i^*	component of velocity
x_i	dimensionless position coordinate relative to moving observer, $\bar{x}_i - \delta_{i1}U_1(\bar{x}_2)t$
$\bar{x}_i$	dimensionless position coordinate, x_i^*/x_0

x_0	characteristic length
α^{m}	defined by eq. (21a)
α^{mn}	defined by eq. (21b)
α^{mnr}	defined by eq. (21c)
β^{m}	defined by eq. (24a)
β^{mn}	defined by eq. (24b)
β^{mnr}	defined by eq. (24c)
β^{mnrs}	defined by eq. (24d)
δ_{ij}	Kronecker delta, 1 for $i = j$, 0 for $i \neq j$
ν	kinematic viscosity
π^{m}	defined by eq. (14a)
π^{mn}	defined by eq. (14b)
ρ	density

Superscripts:

*	dimensional quantity
$-$	averaged velocity or stationary coordinate

REFERENCES

1. Taylor, G. I.; and Green, A. E.: Mechanism of the Production of Small Eddies from Large Ones. Proc. Roy. Soc. (London), Ser. A, vol. 158, no. 895, Feb. 3, 1937, pp. 499-521.
2. Jain, P. C.: Numerical Study of the Navier-Stokes Equations for the Production of Small Eddies from Large Ones. Rep. MRC-TSR-491, Univ. Wisconsin (AD-607807), July 1964.
3. Deissler, Robert G.: Nonlinear Decay of a Disturbance in Unbounded Viscous Fluid. Appl. Sci. Res., vol. 21, Jan. 1970, pp. 393-410. See also NASA TN D-4947, 1968.
4. Parker, Kim H.; and Cavelle, Jean P.: The History of Velocity Waves in a Reacting Fluid. Paper 70-146, AIAA, Jan. 1970.
5. Orszag, Steven A.: Numerical Methods for the Simulation of Turbulence. Phys. Fluids, Suppl. II, vol. 12, no. 12, pt. 2, Dec. 1969, pp. II-250-II-257.
6. Deissler, Robert G.: Effects of Inhomogeneity and of Shear Flow in Weak Turbulent Fields. Phys. Fluids, vol. 4, no. 10, Oct. 1961, pp. 1187-1198.

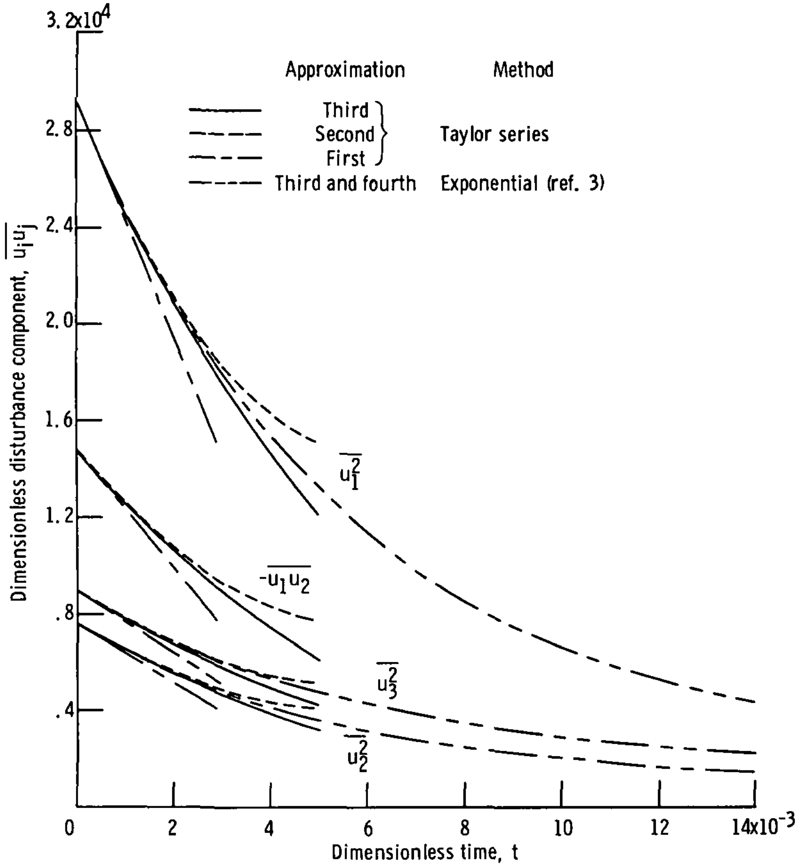


Figure 1. - Comparison of present results for a Taylor series with those from reference 3 for an exponential method for dimensionless velocity gradient of 0. $a_i^1 = 30(-1, 1, -2)$; $a_i^2 = 120(-2, 1, -1)$; $q_i^1 = (2, 4, 1)$; $q_i^2 = 2(1, 4, 2)$.

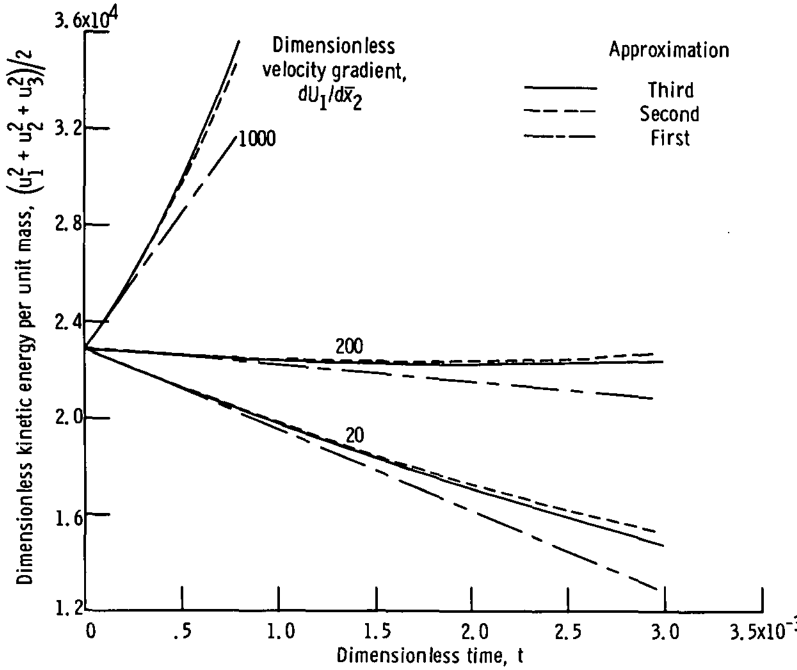


Figure 2. - Effect of mean velocity gradient on evolution of kinetic energy of disturbance. $a_i^1 = 30(-1, 1, -2)$; $a_i^2 = 120(-2, 1, -1)$; $q_i^1 = (2, 4, 1)$; $q_i^2 = 2(1, 4, 2)$.

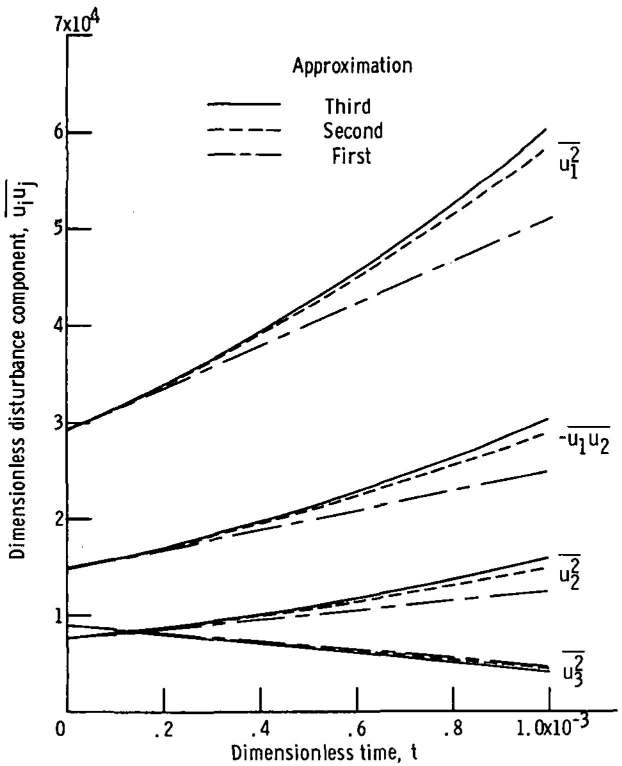


Figure 3. - Evolution of disturbance components for dimensionless velocity gradient of 1000. $a_i^1 = 30(-1, 1, -2)$; $a_i^2 = 120(-2, 1, -1)$; $q_i^1 = (2, 4, 1)$; $q_i^2 = 2(1, 4, 2)$.

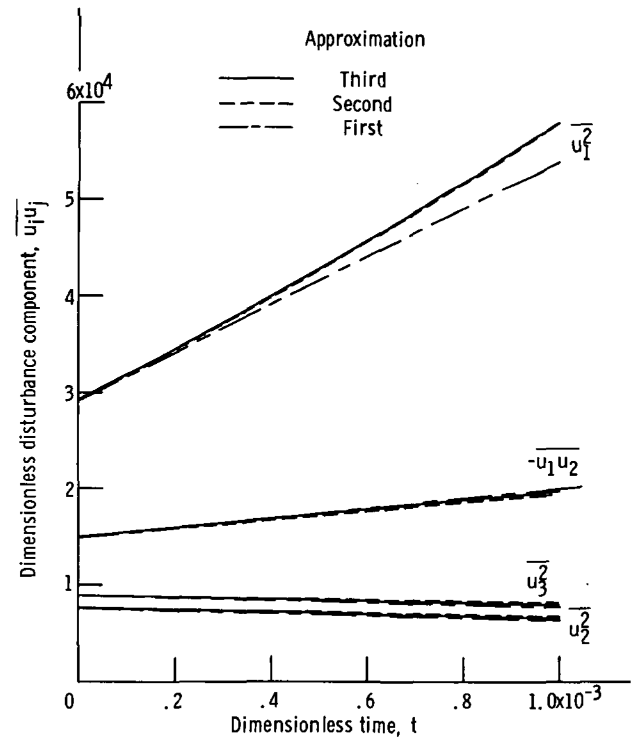


Figure 4. - Evolution of disturbance components for dimensionless velocity gradient of 1000 with pressure terms omitted. $a_i^1 = 30(-1, 1, -2)$; $a_i^2 = 120(-2, 1, -1)$; $q_i^1 = (2, 4, 1)$; $q_i^2 = 2(1, 4, 2)$.

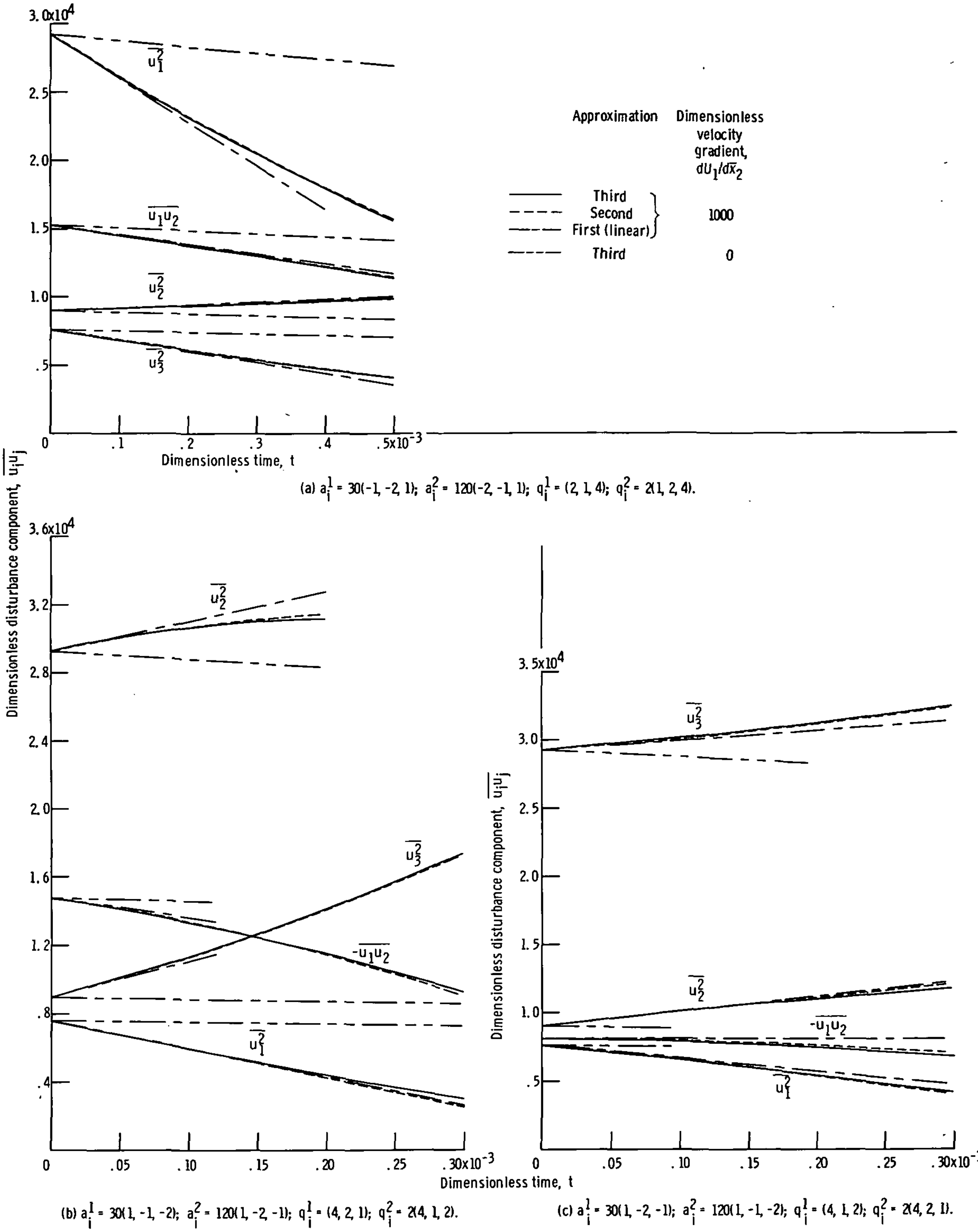
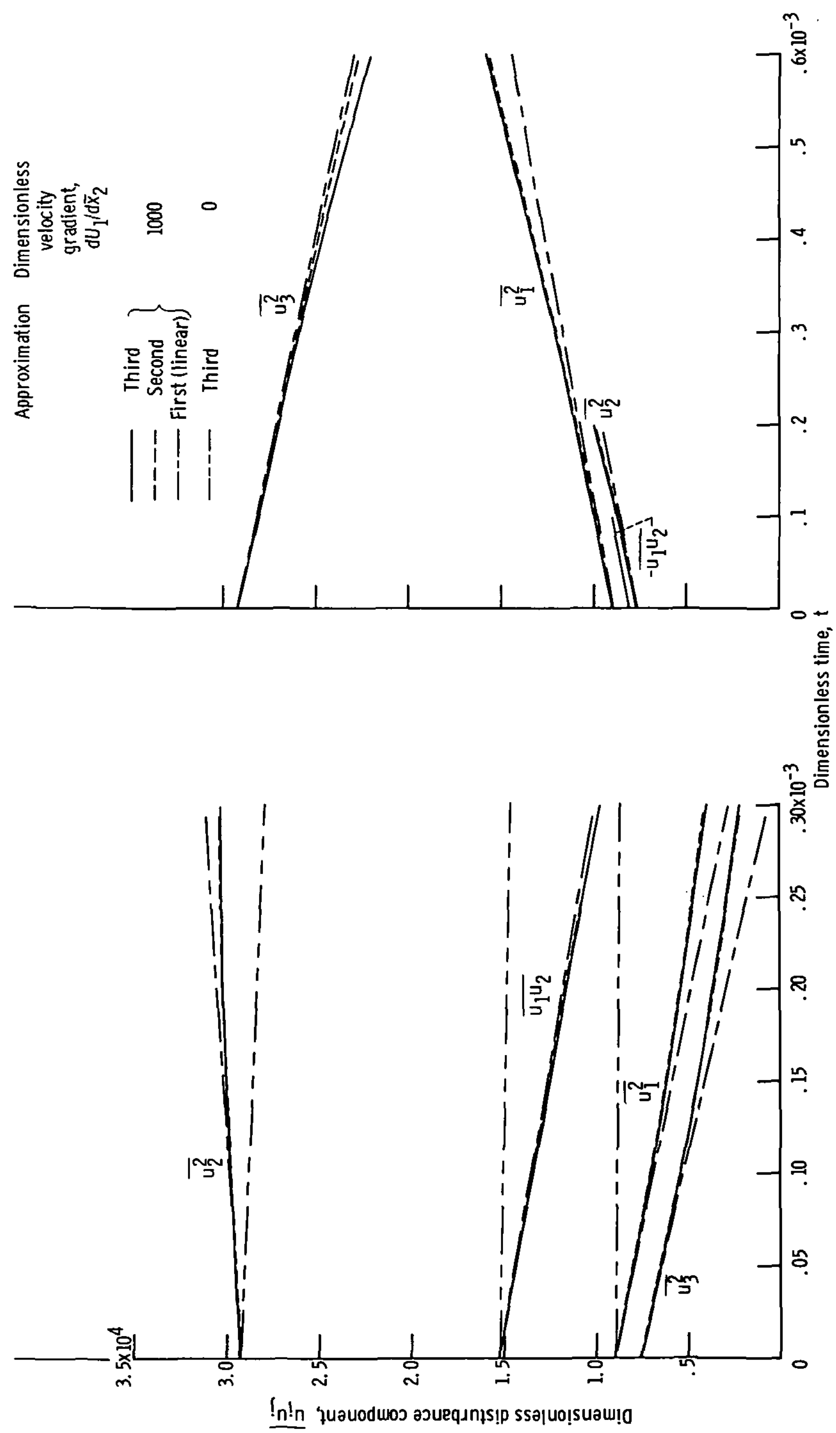


Figure 5. - Evolution of disturbance components for dimensionless velocity gradients of 1000 and 0.



(d) $a_i^1 = 30(-2, -1, 1)$; $a_i^2 = 120(-1, -2, 1)$; $q_i^1 = (1, 2, 4)$; $q_i^2 = 2(2, 1, 4)$.

(e) $a_i^1 = 30(-2, 1, -1)$; $a_i^2 = 120(-1, 1, -2)$; $q_i^1 = (1, 4, 2)$; $q_i^2 = 2(2, 4, 1)$.

Figure 5. - Concluded.

NASA-Langley, 1973 —— 12 E-6857

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons. Also includes conference proceedings with either limited or unlimited distribution.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include final reports of major projects, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546